cannot be excluded. The concentration profiles that are observed (Fig. 3) indicate the significant presence of tracer at the base of the step. The concentration maxima C_m are located in the immediate vicinity of the dividing streamline. We also employed helium (the Schmidt and Prandtl numbers are comparable, despite differences in density). Not withstanding the differences between maximum concentration of the two gases, the concentration profiles tend toward identical values at the wall. But a small significant gap exists between the maximum concentration ordinates obtained for the two injections: helium appears to exhibit greater diffusion toward the outer region. However, the stagnation region – incompressible flow region – practically is not modified by injection. The boundary of this region is naturally very close to the experimental dividing streamline.

Conclusions

Diffusion phenomena in a cavity were investigated by adopting a computation scheme designed for thermal problems. The Navier-Stokes equations with mass transfer were solved in two coupled regions. This model led to the development of a computation program corresponding to the injection of a foreign gas upstream from the near wake, this injection only slightly modifying the dynamic field. The experiments made it possible to determine the existence of dynamic and concentration fields in the base flow.

Comparison of theoretical and experimental results gave rise to the following conclusions:

1) The results are in good agreement with respect to the dynamic field. The gaps that subsist derive from the hypotheses concerning the dividing streamline – boundary between the two regions investigated – made in the simplified model.

2) As for the diffusion field, which quickly reaches a state of equilibrium, a gap exists concerning the mass concentration value at the horizontal wall. The closely comparable results obtained for this value with helium and argon, which have widely differing molecular weights, diffusivity and conductivity, exclude the influence of gravity and thermodiffusion, which are present in the dividing streamline.

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A Near-Optimal Control Law for Pursuit-Evasion Problems Between Two Spacecraft

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Introduction

In Refs. 1 and 2, a general method for generating near optimal feedback solutions to nonlinear zero-sum differential games is presented which allows a player using this method to take advantage of any nonoptimal play of his opponent. In employing this method, a player periodically updates, to first order, his solution to the two-point boundary-value problem (TPBVP) obtained using the necessary conditions for a differential game saddle point solution. At an updating time t_i, this update is accomplished by adjusting the costate vector $\lambda(t_i)$ according to

$$\delta\lambda(t_i) = S(t_i)\delta x(t_i) \tag{1}$$

where $\delta x(t_i)$ is the difference between the actual system state and a reference state indicated by a TPBVP solution. In Ref. 1, $S(t_i)$ is generated by the backward integration of a matrix Riccati differential equation from t_f to t_i, whereas in Ref. 2 it is obtained using the transition matrices for the linearized TPBVP.

The examples presented in these references all are relatively simple with low state dimension. The purpose of this Note is to demonstrate the capability of this method for generating near optimal feedback controls for a more realistic pursuit-evasion problem between two maneuvering space vehicles. The pursuing vehicle employs the near-optimal scheme, whereas the evading vehicle plays various nonoptimal strategies. The payoff is the final range between the two vehicles, the final time is left free, and the game ends when the range rate between the vehicles goes to zero. Coplanar problems are investigated using both the matrix Riccati and transition matrix updating techniques, whereas noncoplanar problems are solved using only the matrix Riccati method.

Problem Statement

The equations describing the motion of a thrusting space vehicle in an inverse-square gravitational field are

$$\dot r = V_r \quad \dot\theta = V_\theta / r \quad \dot\phi = V_\phi/(r\sin\theta)$$

$$\dot V_r = (V_\theta{}^2 + V_\phi{}^2)/r - 1/r^2 + (F/m)\sin\alpha_2$$

$$\dot V_\theta = (V_\phi{}^2\cot\theta - V_r V_\theta)/r + (F/m)\cos\alpha_1\cos\alpha_2$$

$$\dot V_\phi = -(V_r V_\phi + V_\theta V_\phi \cot\theta)/r + (F/m)\sin\alpha_1\cos\alpha_2 \tag{2}$$

where Earth canonical units are used with an Earth-centered spherical coordinate system. The controls are the angles α_1 and α_2, which define the thrust direction. The normalized

Presented as Paper 76-794 at the AIAA/AAS Astrodynamics Conference, San Diego, Calif., Aug. 18-20, 1976; submitted Sept. 15, 1976; revision received May 5, 1977.

Index categories: Analytical and Numerical Methods; Spacecraft Dynamics and Control; Spacecraft Navigation, Guidance, and Flight-Path Control.

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rocket mass m varies according to

$$m = 1 + \dot{m}t \qquad \dot{m} = -F/c \tag{3}$$

where c is the constant effective exhaust velocity. The rocket thrust F, and hence $\dot{m}$, are assumed to be constant.

The problem starts at $t=0$ with the states of the pursuing and evading vehicles specified. The pursuer p then chooses his controls α_{1p} and α_{2p} to minimize the minimum range between the two vehicles, whereas the evader e should choose α_{1e} and α_{2e} to maximize this minimum range. Thus the payoff of this game can be expressed as

$$J = \{r_p{}^2 + r_e{}^2 - 2r_e r_p[\sin\theta_p \sin\theta_e \cos(\phi_p - \phi_e) + \cos\theta_p \cos\theta_e]\}/2 \tag{4}$$

where the subscripts p and e refer to the pursuer and evader, respectively. The final time t_f is free, and the game terminates when $\dot{J}$ (i.e., the range rate) goes to zero with $\ddot{J}>0$.

Equations (2) and (4) are valid for a general noncoplanar problem. If the problem is restricted to be coplanar, the equations of motion and payoff can be obtained by deleting the $\dot{\theta}$ and $\dot{V}_\theta$ equations, and setting $V_\theta = 0$, $\theta = \pi/2$, and $\alpha_1 = \pi/2$ in Eqs. (2) and (4).

The application of the necessary conditions[3] for a differential game saddle-point solution are straightforward and are not repeated here. The result is a TPBVP that is used in generating the near-optimal feedback controls.

Numerical Results

A series of problems with different initial conditions and initial rocket masses, both coplanar and noncoplanar, were simulated on a CDC 6600 computer. In order to illustrate the results, one coplanar and one noncoplanar problem will be presented in detail, with occasional references to other problems.

The coplanar problem assumes the following parameters for both vehicles

$$F = 20{,}000 \text{ lb } (88{,}960 \text{ N})$$

$$M_0 = 900 \text{ slugs } (13{,}131.0 \text{ kg}) \qquad I_{sp} = 300 \text{ sec} \tag{5}$$

The initial conditions in normalized units for the evader are

$$r_e = 1.1 \qquad V_{r_e} = 0 \qquad \phi_e = 1.0$$

$$V_{\phi_e} = 0.9535 \qquad \theta_e = 1.5708 \qquad V_{\theta_e} = 0 \tag{6}$$

For the pursuer, the initial conditions are

$$r_p = 1.02 \qquad V_{r_p} = 0.45 \qquad \phi_p = 1.23$$

$$V_{\phi_p} = -0.34 \qquad \theta_p = 1.5708 \qquad V_{\theta_p} = 0 \tag{7}$$

With optimal play by both vehicles, the minimax range that results is 25.31 naut miles (46.87 km), with a flight time of $t_f = 0.197$ normalized time units (TU), or 158 sec.

Assuming that the pursuer employs the near optimal feedback method with the matrix Riccati updating technique, and that the evader uses constant nonoptimal controls, Fig. 1 shows the final range R_f for different values of nonoptimal evader thrust angle α_{2e}, with a constant updating interval of $\Delta t = 0.012$ TU used by the pursuer. Play by the evader of $\alpha_{2e} = 1$ rad is close to his optimal control, and results in a miss distance of 23.74 naut miles (43.97 km). As α_{2e} moves away from this value, R_f decreases to values close to zero, but then increases as α_{2e} nears values 180 deg from optimal ($\alpha_{2e} = -2$ rad).

At each updating time, a predicted final miss R_p is calculated based on the assumption of optimal play by both vehicles for the remainder of the game. The change in R_p

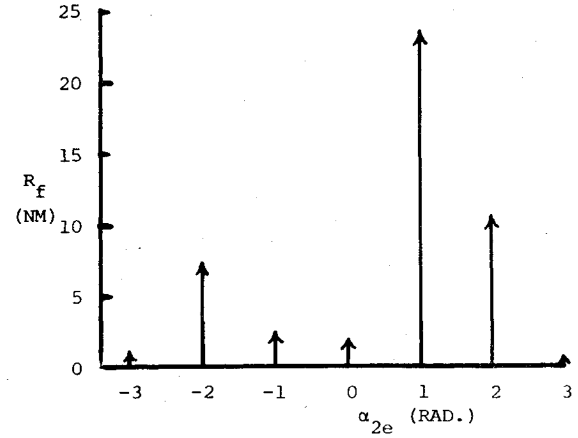

Fig. 1 Final range R_f for coplanar problem with various nonoptimal evader thrust angles α_{2e}.

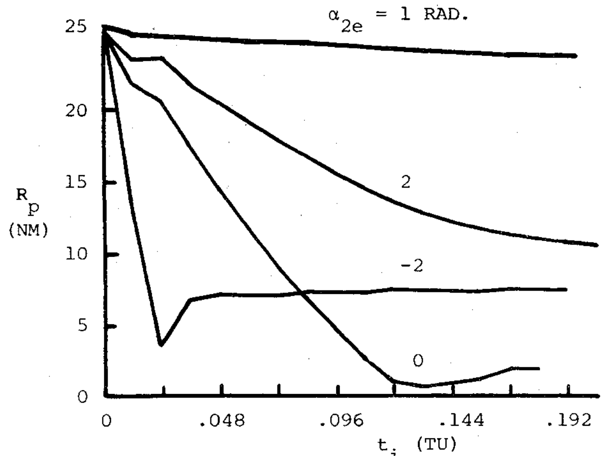

Fig. 2 Predicted final range R_p vs updating time t_i for coplanar problem with various nonoptimal evader thrust angles α_{2e}.

between updates gives a measure of how effectively the control law is able to take advantage of nonoptimal evader play during this interval. Figure 2 shows R_p as a function of updating time t_i for four values of nonoptimal α_{2e}. For $\alpha_{2e} = 1.0$, which is close to optimal, R_p decreases only slightly with increasing t_i. For $\alpha_{2e} = 0$ and -2, R_p is reduced to essentially zero at some point, when nonunique solutions exist for the pursuer to reduce the miss to zero; R_p then increases, indicating a brief failure of the method. For $\alpha_{2e} = -2$, which is close to 180 deg from optimal, R_p is reduced close to zero at the second update, but then increases to about 7 and remains there. The reason for this behavior is that a nonunique control situation occurs at the second update, when the method briefly fails. This results in a state at the third update, from which the nonoptimal evader control $\alpha_{2e} = -2$ is close to optimal, causing R_p to remain almost constant at subsequent updates.

For some other coplanar problems, both the matrix Riccati and transition matrix updating methods produced essentially the same R_p vs t_i history. With a CDC 6600 computer, the maximum time needed for the required computations between updates was about 12.2 sec, with the matrix Riccati updating method and an updating interval of $\Delta t = 0.012$, or 9.7 sec. Although this was not a real-time simulation, no attempt was made to program the problem efficiently, nor was advantage taken of the symmetry of the updating matrix S to reduce the number of differential equations. Thus it should be possible to reduce the computation time well below the updating interval of $\Delta t = 9.7$ sec to get real-time control with the matrix Riccati method. Implementation of the transition matrix

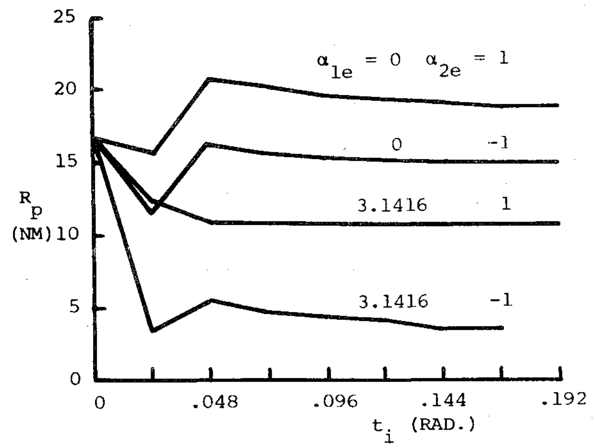

Fig. 3 Predicted final range R_p vs updating time t_i for noncoplanar problem with various nonoptimal evader thrust angles α_{1e} and α_{2e}.

updating method in real-time, however, does not appear feasible because of the large number of equations that must be integrated between updates.

The basic noncoplanar problem has the same rocket thrust and I_{sp} as the coplanar problem, but with initial mass $m(0) =$ 1250 slugs (18,237.5 kg) for both vehicles. The evader's initial conditions are the same, whereas the pursuer's initial conditions are

$$r_p = 1.02 \qquad V_{r_p} = 0.46 \qquad \phi_p = 1.23$$

$$V_{\phi_p} = -0.35 \qquad \theta_p = 1.5 \qquad V_{\theta_p} = 0.38 \tag{8}$$

The final range with optimal play by both vehicles is 16.73 naut miles (30.45 km).

Figure 3 shows the predicted final range R_p as a function of updating time t_i for a number of sets of nonoptimal evader thrust angles. An updating interval of $\Delta t = 0.024$ TU with the matrix Riccati updating method was used in all cases. This R_p tends to oscillate more than in the coplanar problems, especially at the start of the problem, where R_p generally decreases at the first update, increases at the second, after which it slowly decreases. This difficulty at the second update is probably due to instability in the integration of the matrix Riccati equation.

As for possible real-time implementation of this method for noncoplanar problems (with state vector having dimension 12), the maximum computation time required between updates is about 32 sec, compared to an acutal updating interval of 19.4 sec ($\Delta t = 0.024$ TU). As with the coplanar problem, efficient programming and use of the symmetry of the S matrix should allow real-time implementation.

Conclusions

The principal conclusion is that the near optimal feedback control law allows the pursuing spacecraft to take advantage of nonoptimal play of the evader to reduce the final range. With efficient programming, this technique probably can be implemented in real time, using the matrix Riccati updating technique.

However, this technique is not as effective when the predicted final range approaches zero, when nonunique controls occur. One possible solution to this problem is to use the near optimal control law until the predicted final range reaches some specified small value, and then to switch to some other guidance technique, such as proportional navigation.

The instability of the numerical backward integration of the matrix Riccati differential equation is another problem area, especially with the noncoplanar problems. The integration technique used in the simulations was a four point Adams-Bashforth-Moulton predictor-corrector routine that is started by a fifth-order Runge-Kutta method. It is possible that other numerical integration techniques would result in greater stability.

Another problem with this technique is that a solution to the initial TPBVP is required to start the method. Numerically solving a TPBVP is too time consuming to be a viable solution. It may be possible, however, to store the initial costates associated with a number of expected initial states. The method then could be started, either by using the costate vector from the solution closest to the actual initial state, or by interpolating to get an improved estimate of the optimal costate vector.

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A Computationally Fast One-Dimensional Diffusion-Photochemistry Model of SST Wakes

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Introduction

SENSITIVITY studies of models describing global effects of perturbations to the stratospheric ozone layer caused by large-scale operations of aerospace vehicles require consideration of wake photochemistry and diffusion. A computational method applicable to such analysis in the early postvortex phase is presented in this Note. Because of the computational rapidity of the method, sensitivity studies of SST effluent effects upon ozone depletion can be readily accomplished. Comparison with other studies and predictions of some characteristics for global NO_x input are provided.

Derivation and Chemistry

Consider a turbulent plume of cross-sectional area A, which grows with increasing distance x from the origin by turbulent entrainment of material from the surrounding free stratosphere. The conservation of species i mass equation is[1]

$$\mathrm{d}(\rho_i \bar{U} A)/\mathrm{d}x = \mathrm{d}(\rho_{ie} \bar{U} A)/\mathrm{d}x + A\dot{\rho}_i \tag{1}$$

where ρ_i is the area-averaged partial mass density within wake, ρ_{ie} is the constant edge (or ambient) partial density, $\bar{U}$ is the (constant) wind velocity or airspeed in a vehicle-fixed coordinate system, and $\dot{\rho}_i$ is the creation or destruction of species i (mass/volume/time) by chemical transformation. The left-hand side of Eq. (1) represents the rate of change of

Received Jan. 10, 1977; revision received April 20, 1977.

Index categories: Thermochemistry and Chemical Kinetics; Airbreathing Propulsion; Environmental Effects.

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